Mathematics for Economists-2

Module 2, 2022-3

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Course information

Course Website: https://my.nes.ru

Office Hours: By appointment

Class Time: TBA

Room: TBA

TAs: TBA

Course description

This course focuses on recursive dynamic optimization problems used in economic models from a mathematical standpoint, in order to help the students master the tools that are necessary to understand, setup and solve such problems. The first half of the course is dedicated to deterministic optimization problems in discrete time. In the second half, we will discuss uncertainty and take a brief look at optimization problems in continuous time.

Course requirements, grading, and attendance policies

There will be weekly homework assignments (%20) and a final exam (%80). Following the general policy of NES, students are entitled for a make-up exam if they have missed the final with a valid reason or if they have failed in the final. The difficulty of tasks and the grading scheme in the make-up are likely to be different than those in the final.

Course contents

Part 1: Deterministic Dynamic Programing

1. Contraction mappings and their fixed points

2. Bellman equation, existence of a solution, properties of the value functions, applications

3. Euler equations, transversality conditions, a sufficiency theorem

4. A brief look at dynamic stability

Part 2: Uncertainty

5. Markov chains and asymptotic stationarity

6. Stochastic dynamic programing, applications

Part 3: Continuous Time

7. A brief introduction to optimal control in continuous time

Description of the course methodology

If external conditions permit, the instructor will use the traditional methods in a classroom (i.e., a whiteboard, a marker and verbal discussions). Otherwise, we will have online classes. In either case, students are encouraged to participate in lectures with questions and comments. Attendance is particularly important because the lecture notes will be self-contained, and there is no required textbook.

Course materials

Required Textbook:

There is no required textbook. The lecture notes will be self-contained.

Additional/Optional reading:

Ljungqvist, L. and Sargent, T., *Recursive Macroeconomic Theory*, MIT Press, 2004.

Stokey, N.L. and Lucas, R.E., with Prescott, E.C., *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.

Chiang, A.C., *Elements of Dynamic Optimization*, Waveland Press, 1992.

Notes: Stokey and Lucas is an excellent reference for the foundations of discrete-time dynamic programing, though it is not so suitable for self-study unless you are somewhat familiar with functional analysis. Ljungqvist and Sargent is more accessible, focuses on the essential aspects, but does not elaborate on some important issues. Chiang offers a nice exposition of alternative approaches to continuous time optimization problems with a number of economic examples.

Academic integrity policy

Cheating, plagiarism, and any other violations of academic ethics at NES are not tolerated.

Sample tasks for course evaluation

Consider the following problem: Given $k_0 \in \mathbb{R}_+$,

$$\max_{(c_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{s.t.} \quad c_t \le k_t, \qquad k_{t+1} = k_t - c_t, \qquad \text{and } c_t, k_t \in \mathbb{R}_+ \quad \text{for } t = 0, 1, \ldots \tag{SP}$$

Here, k_t and c_t represent capital stock and consumption at time t, respectively, while $\beta \in (0,1)$ is the discount factor. The saving $k_t - c_t$ becomes the capital stock at time $t+1$.

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. For any function $v : \mathbb{R}_+ \to \overline{\mathbb{R}}$, define a further function $Tv : \mathbb{R}_+ \to \overline{\mathbb{R}}$ by

$$Tv(k) = \sup_{c \in [0,k]} \ln(c) + \beta v(k - c) \quad \forall k \in \mathbb{R}_+. \tag{1}$$

The *Bellman equation* refers to the following functional equation:

$$v(k) = Tv(k) \quad \forall k \in \mathbb{R}_+.$$

a) Use the guess and verify method to find a solution $v : \mathbb{R}_+ \to \mathbb{R}$ to the Bellman equation.

b) Define a function v^0 by $v^0(k) = \frac{\ln k}{1-\beta}$ for every $k \in \mathbb{R}_+$. Inductively, let $v^n = Tv^{n-1}$ for every $n \ge 1$. Using the FOC for the problem (1), show that $\lim_n v^n(k) = v(k)$ for every $k \in \mathbb{R}_+$, where v stands for the function you have found in part (a). (*Hint.* There exists a constant D such that $v^n(k) = v^0(k) + D\sum_{l=0}^{n-1} \beta^l$ for every $k \in \mathbb{R}_+$.)

c) Conclude that for any $k = k_0$, the maximum value in (SP) equals $v(k)$. Verify all assumptions in the related theorem that we have covered in class.

d) Write down the Euler equations in terms of the consumption levels.

e) Show that, given the constraints in (SP) and the Euler equations, the transversality condition holds iff $\lim_t k_t = 0$ iff $\sum_{t=0}^{\infty} c_t = k_0$.

f) Conclude that the optimal consumption sequence implied by the Bellman equation is also induced by the Euler equations and the transversality condition.